

GENERALIZED SIMILARITY LAWS IN FLOWS PAST SOLID BODIES

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We show that when the localizability law prevails [1], i. e. in the case when the momentum stream at the surface of a body depends basically on the local angle between the normal to the surface and the direction of the flight velocity (e. g. hypersonic Newtonian gas flow, rarefied gas flow, influence of light, etc.), generalized similarity laws can be established connecting the aerodynamic characteristics of the nonaffine-similar bodies in gas flows of different modes (e. g. the reference body is in a Newtonian flow and the complementary body is in a free-molecular flow of rarefied gas). Methods of constructing the complementary bodies are worked out and examples of application of the proposed similarity laws are given. A particular example of such a law for a (plane-parallel) flow past a profile under certain additional restrictions was considered under the assumptions of the localizability law in [2], and for a Newtonian hypersonic gas flow in [3].

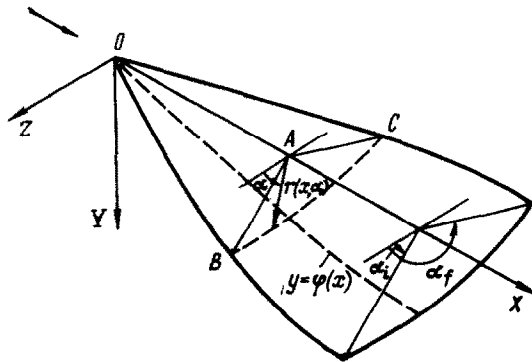


Fig. 1

1. Statement of the problem. We consider a flow past a solid body with a homothetic property [4] at zero angle of attack under the conditions of "the localizability law". The equation of the surface of the body (Fig. 1) is given in the form

$$r(x, \alpha) = \frac{\rho(\alpha)^i}{\rho(\pi/2)} \varphi(x) = \rho_*(\alpha) \varphi(x), \quad \varphi(0) = 0, \quad \alpha_i \leq \alpha \leq \alpha_f, \quad 0 \leq x \leq x_f \quad (1.1)$$

where the functions $\rho(\alpha)$ and $\varphi(x)$ characterize, respectively, the lines of intersection of the surface of the body with a certain plane perpendicular to the OX -axis, and with the XOY -plane. The expressions for the drag coefficient c_x and lift coefficient c_y with the localizability law prevailing [1], are considerably simplified when the integration with respect to α and x can be carried out separately. This occurs in the case of slender bodies ($\rho_*^2 \varphi^2 \ll 1$), while for an arbitrary function $\varphi(x)$ it takes place only in

the case when the function $\rho_*(\alpha)$ satisfies the equation

$$1/\rho_*^2 + \rho_*'^2/\rho_*^4 = C = \text{const} \quad (1.2)$$

i. e. when the boundary of intersection with the ABC planes perpendicular to the OX -axis consists of arcs of circles with the center at A , and straight line segments (these cases were mentioned in [5], where a hypersonic Newtonian flow was considered). Thus this class includes, in particular, semisolids of revolution and solids with polygonal transverse cross sections.

Assuming that the function $\rho_*(\alpha)$ is given, we can write the expressions for the characteristics c_x and c_y in the form

$$c_x = P_1, \quad c_y = P_2, \quad P_i = \int_0^1 \varphi(\bar{x})^{\nu} f_i(\theta) d\bar{x}, \quad \bar{x} = \frac{x}{x_j}, \quad \theta = \arctg \frac{d\varphi(\bar{x})}{d\bar{x}} \quad (1.3)$$

where the functions $f_i(\theta)$ are determined by the character of the flow and by the form of the transverse cross section, and $\nu = 1$.

We note that in the case of a flow past a profile the lower boundary of which is determined by the function $\varphi(x)$ and the upper boundary is the OX -axis, the formulas for the coefficients c_x and c_y are obtained from (1.3) with $\nu = 0$ [2]. For the solids the characteristics of which are represented by (1.3), transformations (generally nonaffine) can be found, which transform the reference solids into the complementary solids of the same class, and the similarity laws derived which can be used to compute the aerodynamic characteristics of the complementary solid from those of the reference solid.

2. Complementary solids. In what follows, the quantities $x^{(\nu)}$ and $y^{(\nu)}$ ($\nu = 0, 1$) will be assumed dimensionless (related to $x_j^{(0)}$). Let the form of the reference solid be given by the function $\rho(\theta)$ and the equation of intersection of the body surface with the plane $Z = 0$

$$y^{(0)} = \varphi(x^{(0)}), \quad \varphi(0) = 0, \quad \varphi \geq 0, \quad \varphi' \geq 0 \quad \text{for } 0 \leq x^{(0)} \leq 1$$

The equation $y^{(0)} = \varphi(x^{(0)})$ can be written parametrically in the form

$$\begin{aligned} x^{(0)} &= F^{-1}(\xi), \quad y^{(0)} = \varphi[F^{-1}(\xi)] \\ \xi &= \int_0^{x^{(0)}} \varphi(t)^{\nu_0} \Phi_0[\theta^{(0)}(t)] dt = F(x^{(0)}), \quad \theta^{(0)}(t) = \arctg \varphi'(t) \end{aligned} \quad (2.1)$$

Here φ and Φ_i are sufficiently smooth functions nonnegative on $(0, \pi/2)$ which define the transformation, $\nu_j = 0$ in the case of a flow past a profile and $\nu_j = 1$ in the case of a flow past a homothetic body.

If the form of the transverse cross section of the body obtained by the transformation is known, then the form of the body surface will be defined, if the equations $y^{(1)} = y^{(1)}(\xi)$ and $x^{(1)} = x^{(1)}(\xi)$ are given. The functions $y^{(1)}(\xi)$ and $x^{(1)}(\xi)$ are found from

$$y^{(1)\nu_1} x^{(1)\nu_2} \Phi_1(\theta^{(1)}) = 1, \quad y^{(1)} = x^{(1)} \operatorname{tg} \theta^{(1)}, \quad \theta^{(1)}(\xi) = \chi \left(\arctg \frac{y^{(0)}}{x^{(0)}} \right)$$

and have the form

$$\begin{aligned} x^{(1)} &= (\nu_1 + 1)^{-\nu_1/(\nu_1+1)} \int_0^{x^{(0)}} Q(\theta) \varphi(t)^{\nu_0} U(t)^{-\nu_1/(\nu_1+1)} dt, \\ y^{(1)} &= [(\nu_1 + 1) U(x^{(0)})]^{-1/(\nu_1+1)} \end{aligned} \quad (2.2)$$

$$U(x) = \int_0^x Q(\theta) \operatorname{tg}[\chi(\theta)] \varphi(s)^{\nu_0} ds, \quad \theta = \theta^{(0)}, \quad Q(\theta) = \frac{\Phi_0(\theta)}{\Phi_1[\psi(\theta)]} \tag{2.3}$$

where $x^{(0)}$ is chosen as the parameter.

3. Similarity relations. The aerodynamic characteristics of the reference (superscript ⁰) and complementary (superscript ¹) bodies can be written, after changing the variable of integration to ξ , in the form

$$P_i^{(0)} = \int_0^{\xi_j} \frac{f_i^{(0)}(\theta) d\xi}{\Phi_0(\theta)}, \quad P_j^{(1)} = \int_0^{\xi_j} \frac{f_j^{(1)}[\psi(\theta)] d\xi}{\Phi_1[\chi(\theta)]}, \quad \xi_j = F(1), \quad i = 1, 2, \dots, n_0, \tag{3.1}$$

$j = 1, 2, \dots, n_1$

where n_0 and n_1 denote the respective number of the characteristics considered for the reference and the complementary body. We obviously have

$$\int_0^{\xi_j} \frac{\operatorname{tg} \theta d\xi}{\Phi_0(\theta)} = \frac{1}{\nu_0 + 1} y_j^{(0)\nu_0+1}, \quad \int_0^{\xi_j} \frac{\operatorname{tg}[\chi(\theta)] d\xi}{\Phi_1[\chi(\theta)]} = \frac{1}{\nu_1 + 1} y_j^{(1)\nu_1+1} \tag{3.2}$$

Let us assume that a set of constants $a_k^{(q)}$ not all equal to zero exists such that the following identity holds:

$$\sum_{i=0}^{n_0} a_i^{(0)} f_i^{(0)}(\theta) + Q(\theta) \sum_{j=0}^{n_1} a_j^{(1)} f_j^{(1)}[\psi(\theta)] = 0, \quad f_0^{(0)} = \theta, \quad f_0^{(1)} = \operatorname{tg}[\psi(\theta)] \tag{3.3}$$

Then, dividing (3.3) by $\Phi_0(\theta)$, making use of (2.2), (2.3) and (3.2) and integrating in ξ from 0 to ξ_j , we obtain a relation connecting the aerodynamic characteristics of the reference and the complementary body independent of the form of the reference body

$$\sum_{i=1}^{n_0} a_i^{(0)} P_i^{(0)} + \frac{1}{\nu_0 + 1} y_j^{(0)\nu_0+1} + \sum_{j=1}^{n_1} a_j^{(1)} P_j^{(1)} + \frac{1}{\nu_1 + 1} y_j^{(1)\nu_1+1} = 0 \tag{3.4}$$

Since several linearly independent sets of quantities $a_k^{(q)}$ may exist, we can have several different relations of the type (3.4). Thus the transformation is defined by two functions

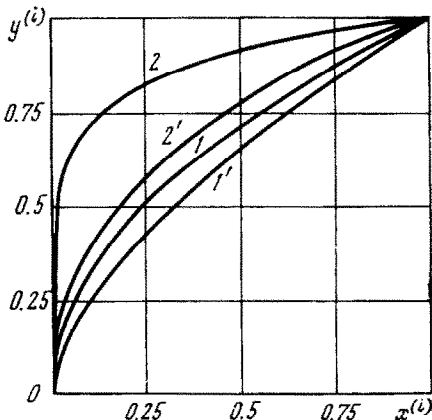


Fig. 2

Q and ψ . We note that the modes of flows past the reference and the complementary body can be different. When the complementary body has a geometric shape ($\nu_1 = 1$), the function $\rho^{(1)}$ must be given.

4. Example. Let the reference and the complementary body be either in a hypersonic Newtonian flow, or in a free molecular rarefied gas flow, the flow modes not being necessarily the same for both bodies. We assume that the reference body is a semisolid of revolution bounded from above by the plane $Y = 0$. The complementary body will be sought in the class of geometric bodies the transverse cross section of which, parallel to

the YOZ-plane, has the form of a triangle with the vertices $(y, z) = (0, 1), (1, 0), (0, -1)$. The expressions given in [1] for the aerodynamic characteristics lead to the formulas

$$P_1^{(0)} = \frac{c_x^{(1)} - 0.5B_1^{(0)}y_f^{(0)2}}{\pi(A_2^{(0)} - B_1^{(0)})} = \int_0^1 \varphi^{(0)}(x^{(0)}) \frac{\operatorname{tg}^3 \theta^{(0)} dx^{(0)}}{1 + \operatorname{tg}^2 \theta^{(0)}}$$

$$P_2^{(0)} = -\frac{c_y^{(0)}}{2(A_2^{(0)} - B_1^{(0)})} = \int_0^1 \varphi^{(0)}(x^{(0)}) \frac{\operatorname{tg}^2 \theta^{(0)} dx^{(0)}}{1 + \operatorname{tg}^2 \theta^{(0)}}$$

$$P_1^{(1)} = \frac{c_x^{(1)} - B_1^{(1)}y_f^{(1)2}}{2(A_2^{(1)} - B_1^{(1)})} = \int_0^1 \varphi^{(1)}(x^{(1)}) \frac{\operatorname{tg}^3 \theta^{(1)} dx^{(1)}}{2 + \operatorname{tg}^2 \theta^{(1)}}$$

$$P_2^{(1)} = \frac{c_y^{(1)}}{2(A_2^{(1)} - B_1^{(1)})} = \int_0^1 \varphi^{(1)}(x^{(1)}) \frac{\operatorname{tg}^2 \theta^{(1)} dx^{(1)}}{2 + \operatorname{tg}^2 \theta^{(1)}}$$

Let us define the transformation by the functions

$$\chi(\theta) = \operatorname{arctg}(\sqrt{2} \operatorname{tg} \theta), \quad Q(\theta) = \sqrt{2} / \operatorname{tg} \theta \quad (4.1)$$

The identity (3.3) and (4.1) together yield two linearly independent sets of coefficients

$$a_0^{(0)} = a_1^{(0)} = a_0^{(1)} = a_2^{(1)} = 0, \quad a_1^{(1)} = 1, \quad a_2^{(0)} = -2$$

$$a_0^{(1)} = a_2^{(0)} = a_1^{(1)} = 0, \quad a_2^{(1)} = -1, \quad a_0^{(0)} = -\sqrt{2}, \quad a_1^{(0)} = \sqrt{2}$$

The similarity relations (3.4) have the form

$$c_x^{(1)} = B_1^{(1)}y_f^{(1)2} - 2Ec_y^{(0)}, \quad c_y^{(1)} = \frac{\sqrt{2}}{\pi} E(2c_x^{(0)} - \pi B_1^{(0)}y_f^{(0)2}) - \sqrt{2}y_f^{(0)2}(A_2^{(1)} - B_1^{(1)})$$

$$E = (A_2^{(1)} - B_1^{(1)}) / (A_2^{(0)} - B_1^{(0)})$$

Using the formulas (2.2) and (4.1) we write the equation of the complementary contour in the parametric form

$$x^{(1)} = \frac{1}{\sqrt{2}} \int_0^{x^{(0)}} \frac{\varphi^{(0)}(t) dt}{\varphi^{(0)}(t)} \left(\int_0^t \varphi^{(0)}(u) du \right)^{1/2}, \quad y^{(1)} = 2 \left(\int_0^{x^{(0)}} \varphi^{(0)}(t) dt \right)^{1/2}$$

When $y^{(0)} = ax^{(0)n}$, the equations of the complementary contour become

$$y^{(1)} = bx^{(1)(n+1)/(3-n)}, \quad b = a^{(5+n)/2(3-n)} \frac{2}{\sqrt{n+1}} x_f^{(1)(n+1)/(n-3)},$$

$$x_f^{(1)} = \frac{\sqrt{2}(n+1)}{n(3-n)}$$

Figure 2 depicts the reference and the complementary (denoted by a prime) contours for the cases $n = 1/2$ (1) and $n = 1/7$ (2).

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**PROOF OF THE NUMERICAL METHOD OF "DISCRETE VORTICES"
FOR SOLVING SINGULAR INTEGRAL EQUATIONS**

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Proof is given of the convergence of the numerical method of discrete vortices (see, e. g., [1-4]) in the solution of real one-dimensional singular integral equations of the first kind. It is shown that in the class of functions which are unlimited at one end of the integration segment and limited at the other, there exists a unique solution for which the Chaplygin - Joukowski condition is satisfied.

1. Statement of problem. Computation scheme. We consider the real one-dimensional singular integral equation (SIE) of the first kind

$$\int_a^b \gamma(x) \frac{K(x_0, x)}{x - x_0} dx = f(x_0) \quad (1.1)$$

with following conditions (conditions A): $f(x_0)$ satisfies Hölder's condition [5] with exponent α , $H(\alpha)$, for $a < x_0 < b$; $H(x_0, x)$ satisfies the condition $H(\alpha)$ with respect to x_0 and x in the region $a \leq x_0, x \leq b$; $\gamma(x)$ is the unknown function to be determined in the class of functions which are limited for $x = b$ and unlimited for $x = a$. For $x = a$ function $\gamma(x)$ tends to infinity of order ν ($0 < \nu < 1$) and can be, consequently, represented in the form $\gamma(x) = \varphi(x)(x - a)^{-\nu}$, where $\varphi(x)$ satisfies the condition $H(\alpha)$ for $a \leq x \leq b$.

The computation scheme of the considered method consists of dividing segment $[a, b]$ into n equal parts of length h on each of which at a distance of $1/4 h$ from their left-hand end are marked computation points x_i at which values of the sought function $\gamma(x_i)$ are calculated. At the same distance from the right-hand end are located check points x_{0j} ($i, j=1, 2, \dots, n$) at which boundary conditions are satisfied. Thus each check point x_{0j} lies in the middle between adjacent computation points x_j and x_{j+1} , except point x_{0n} which divides segment $[x_n, b]$ in a 2 : 1 ratio.

The numerical method of discrete vortices consists of substituting for the SIE (1.1) of a system of n linear algebraic equations